M.Math. IInd year
First semestral examination 2006
Commutative Algebra — B.Sury
Answer 6 questions — Be brief

Q 1.
Let M be a finitely generated module over a Noetherian local ring A. Prove that M is free if it is flat.

OR

Show that an A- module M is injective if, for each ideal I of A, any A-module homomorphism from I to M extends to the whole of A.

Q 2.
If A is a ring over which every projective module is free, show that the only idempotents (that is, elements $e \in A$ satisfying $e^2 = e$) are 0 and 1.

Q 3.
Prove that a faithfully flat module is faithful and flat but give an example to show that the converse may not hold.

Q 4.
For a field K, consider the ring $A = K[X, Y]/(X^3 - Y^2)$. Prove that A is a domain and decide whether it is integrally closed.

OR

Let A be a domain. Prove that it is integrally closed if and only if $A[X]/(f)$ is a domain for each monic irreducible polynomial $f \in A[X]$.

Q 5.
If M is a finitely generated module over a Noetherian ring, show that the set of zero divisors of M equals $\bigcup\{P : P \in Ass(M)\}$.

OR

In the ring $A = \prod_{\mathbf{N}} \mathbf{R}$, consider for each $n \in \mathbf{N}$,

$$m_n := \{f : \mathbf{N} \to \mathbf{R}, f(n) = 0\}.$$

Show that m_n is a maximal ideal for every n and that $I := \oplus_{\mathbf{N}}\mathbf{R}$ is an ideal contained in $\bigcup_{n \geq 1} m_n$ but not contained in m_n for any n.

Q 6.
Show that in the ring $A = \mathbf{Z}[X]$, the ideal $I = (X, 4)$ is $\mathcal{M}$-primary, where $\mathcal{M} = (X, 2)$, but that I is not a power of $\mathcal{M}$.

Q 7.
Find all the prime ideals of $\mathbf{Z}[i]$ which lie over : (i) 2, (ii) 3, (iii) 5 in $\mathbf{Z}$.

Q 8.
For a field K, consider the subalgebra A of $K[X,Y]$ generated by the monomials $X, X^2Y, X^3Y^2, \cdots$ Prove that $A[XY]$ is contained in a finitely generated A-module but that XY is not integral over A.

Q 9.
Let A be the ring of infinitely differentiable functions from $\mathbf{R}$ to itself. Let $\mathcal{M}$ be the maximal ideal consisting of all the functions which vanish at 0. Find (with proof) a non-zero function which belongs to the intersection $\bigcap_n \mathcal{M}^n$.

OR

Let K be any field of characteristic zero. Consider the formal power series $e(X) := \sum_{r=0}^{\infty} \frac{X^r}{r!}$ as an element of the quotient field $K((X))$. Prove that X and $e(X)$ are algebraically independent transcendental elements over K.
Hint : You may use the fact that the formal derivative f' of an element $f = \sum_{r=-\infty}^{\infty} a_r X^r$ of $K((X))$ cannot be zero if $f \notin K$.

Q 10.
Show that a valuation ring A must be integrally closed. Further, prove that if A is also Noetherian, then it must be a PID.

OR

Prove that a Noetherian ring satisfies the descending chain condition on prime ideals.
Hint : You may use the dimension theorem.